STABILIZATION OF A CART AND DOUBLE-PENDULUM SYSTEM

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ABSTRACT. This Master's Plan B report for the University of Hawai'i at Mānoa is the result of Daniel Langdon's examination of methods of modeling and stabilizing a cart and double-pendulum system against small disturbances. The Euler-Lagrange Equations, a fundamental result of the calculus of variations, combined with the Principle of Least Action and the Lagrange D'Alembert Principle are used to describe the equations of motion for a cart and double-pendulum system in terms of the kinetic and potential energy of the system, which is in turn described in terms of the positions and velocities of the cart and two pendulum bobs. Theorems from differential equations are combined with a linear approximation of the equations of motion and the notion of feedback to compute an algorithm for stabilization, whose action of stabilization against a sample small disturbance is demonstrated.

1. Introduction

For my master's research, I have explored the field of control theory by searching for controls that stabilize a cart and double-pendulum system in the up unstable equilibrium with a single control acting only on the cart. Stabilizing the cart and double-pendulum system is not the end-goal; the purpose of the system is to provide a tool for learning about control theory. Control theory is an important field in applied mathematics with applications in many fields including physics, economics, computer science, and artificial intelligence.

For example, consider the problem of controlling the thrust of a rocket engine as it launches a satellite into orbit. The engine must keep the entire rocket stabilized by applying force only at the bottom of the rocket. This problem, and others like it, have motivated the study of so-called *underactuated* mechanical systems; machines whose mechanics comprise movement over many degrees of freedom, or dimensions; but are actuated by a controller with fewer degrees of freedom. The cart and double-pendulum is a three dimensional system that I stabilize using a controller with one degree of freedom. This problem, in particular, has been studied in connection with rocket science [7]. In practice, a mechanical system may model something with many, many, degrees of freedom, but we study only a few, called the *relevant degrees of freedom* [5]. A mechanical system that models the price of a publicly traded stock would be one such example.

You can see that the relatively simple problem of stabilizing a cart and double-pendulum with a one-dimensional control is not easy. One can intuitively imagine how to stabilize a cart and

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single-pendulum system; but intuitively, it is not even clear that it is possible to stabilize a cart and double pendulum with one control. I confess that when I first examined the problem, the first thing I did was to look for a way to prove that it is impossible. It has been shown, in theory, that any finite number of pendulums may be balanced upon one another using a single control, and engineers have balanced four in practice. [1]

2. A Simple Example

I would like to begin with a simple example to illustrate some of the mathematical techniques I use to stabilize the cart and double-pendulum system. I will begin by modeling the motion of the cart without the double-pendulum attached. Even though it is intuitively 'obvious' how to control a cart rolling back and forth in a straight line, there is more to it than meets the eye. I will describe the motion formally.

The cart is constrained to move in a straight line. We cause the cart to accelerate by applying an acceleration, $u(t)$. Assuming that there is no force on the cart due to gravity, I model the motion of the cart with the second order differential equation

$$\frac{d^2}{dt^2}x(t) = u(t). \tag{2.1}$$

When I stabilize the cart and double-pendulum, I will want to keep the system in an unstable-equilibrium. That is, a position from which it will not change due to potential forces, such as gravity, but when disturbed, will move away from the unstable equilibrium. The input force $u(t)$ will be used to bring the system back to the unstable equilibrium. For this simple system with no pendulum arms, $x = 0$ will suffice as an equilibrium. This system is also easier because we don't have to worry about gravity pushing the cart away from the equilibrium point. Let's look at how we can stabilize this system at $x = 0$ against a disturbance.

Suppose the system's state has been disturbed from $x = 0$ to $x = 1$. In a more realistic situation, the cart would also probably have non-zero velocity, but let's ignore that for now. What kind of control u will bring the system back to $x = 0$ and keep it there? Let's look at some *feedback* models. Feedback means that $u(t)$ is a function of $x(t)$, the state of the system. Let's try the following control:

$$u(t) = \begin{cases} 1 & \text{for} \quad x(t) < 0 \\ -1 & \text{for} \quad x(t) \geq 0 \end{cases}$$

Basically, this control pushes toward the equilibrium point with a constant magnitude. This control doesn't stabilize the system. Although we are constantly pushing the cart toward $x = 0$, the configuration of the system oscillates between $x = 1$ and $x = -1$ forever. Let's try another control. Let's push the cart toward $x = 0$, but this time, let the magnitude of the acceleration be proportionate to the distance between the cart and the point $x = 0$. I define this control by

$$u(t) = -x(t) \tag{2.2}$$

When I substitute and solve the resulting differential equation, I get $x(t) = x(0)\sin(t) + \frac{d}{dt}x(t)|_{t=0}\cos(t)$. Therefore, it is clear that this control oscillates the cart about $x = 0$ forever as well. Any student who passed differential equations can tell by now that adding a damping force to the controller will stabilize the system. However, I will rigorously derive this controller the same I way that I derive the feedback controllers that I will use to stabilize the cart and double-pendulum system.

When I model the cart and double-pendulum, I will want to model it as a system of first-order differential equations. I do the same here. Therefore, I make the substitution $\frac{d}{dt}x(t) = v_x$ to obtain the system

$$\begin{aligned} \frac{d}{dt}v_x(t) &= u(t) \\ \frac{d}{dt}x(t) &= v_x(t). \end{aligned} \tag{2.3}$$

Please take note that although $v_x(t)$ is the velocity of the cart, I treat it here as a separate variable related to $x(t)$ by a differential equation. Here are the equations in the vector form that I will use for the cart and double-pendulum

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \tag{2.4}$$

$$\mathbf{x}(t) = \begin{pmatrix} v_x(t) \\ x(t) \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{u}(t) = \begin{pmatrix} u(t) \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Take note of the fact that $\mathbf{u}(t)$ is a column vector with only one element. Next, I make the linear feedback substitution $\mathbf{u}(t) = \mathbf{F}\,\mathbf{x}(t)$ where $\mathbf{F} = \begin{pmatrix} \mathbf{F}_1, & \mathbf{F}_2 \end{pmatrix}$ is a row-vector to be determined later. Equation (2.4) becomes

$$\frac{d}{dt}\mathbf{x}(t) = \left(\mathbf{A} + \mathbf{B}\,\mathbf{F}\right)\mathbf{x}(t) \tag{2.5}$$

The matrix stability theorem, (10.5) says that $\mathbf{x}(t)$ goes asymptotically to 0 if the eigenvalues of the matrix $\mathbf{A} + \mathbf{B}\,\mathbf{F}$ all have negative real part. I calculate that the eigenvalues of this matrix are $\frac{1}{2}\,(F_1 + \sqrt{F_1^2 + 4\,F_2})$ and $\frac{1}{2}\,(F_1 - \sqrt{F_1^2 + 4\,F_2})$. Therefore, let $\mathbf{F} = \begin{pmatrix} -1, & -1 \end{pmatrix}$. Then $u(t) =$

$-v_x(t) - x(t)$ and the matrix $\mathbf{A} + \mathbf{B}\,\mathbf{F}$ has eigenvalues with negative real part. I calculate the equation of motion for the cart with this control to be

$$x(t) = \frac{\sqrt{3}}{3}\,(x(0) + 2\,v_x(0))\,e^{\frac{-t}{2}}\,\sin(\frac{\sqrt{3}\,t}{2}) + x(0)\,e^{\frac{-t}{2}}\,\cos(\frac{\sqrt{3}t}{2}) \tag{2.6}$$

We can see now that the cart asymptotically approaches the configuration $x = 0$ from any position with the control $u(t) = -\frac{d}{dt}x(t) - x(t)$. Now, let's move on to the business of describing the equations of motion for the cart and double-pendulum system and a controller to stabilize it in the up equilibrium.

3. The Cart and Double-Pendulum System

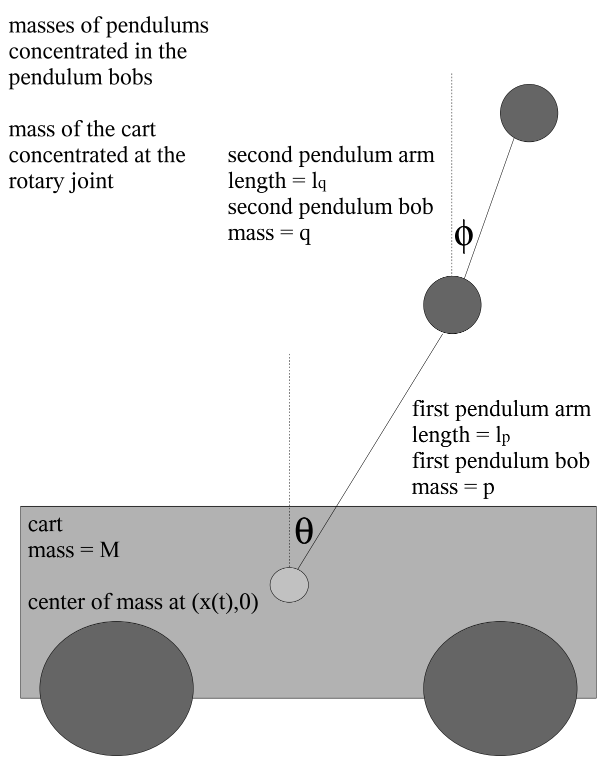


FIGURE A

First of all, the only difference between the model of the cart and single-pendulum system and the cart and double-pendulum system is the absence of the second pendulum bob in the cart and single-pendulum system model. In both systems, I use the same names for the constants that both systems have in common.

I model the cart and double-pendulum system as a set of three point masses. This simple model avoids the complicated multiple integrals associated with modeling the cart and pendulum bobs as bodies with shape. [4]

All motion is constrained to happen in a two-dimensional Euclidean space. The cart is constrained to move in a straight line. The first pendulum bob is constrained to move in a circle around the center of mass of the cart. The second pendulum bob is constrained to move in a circle around the center of mass of the first pendulum bob.

Choose a point of origin on the cart's line of motion. Let x be the displacement of the cart along the line relative to the origin. Let the straight up position of the first pendulum serve as a reference for angle θ, the angle of the first pendulum. Likewise, let the straight up position, perpendicular to the motion of the cart, be the reference for angle ϕ, the angle of the second pendulum. Let θ and ϕ increase as the pendulum arms swing clockwise. Let g be the constant of the gravitational acceleration that points in the direction opposite of the straight up position. Let u be the acceleration of the cart that I control. I call the lengths of the first and second pendulum bobs l_p and l_q, respectively. I call the masses p and q, respectively. I call the mass of the cart M.

4. Machines

Although it is not entirely necessary to introduce the terminology of a *machine*, or alternatively, a *mechanical system* in order to discuss the stabilization of the cart and double-pendulum system, I would like to introduce some control theory framework to illustrate how this problem fits into the field. I use terminology as in [2]

Definition 4.1. A **machine** Σ is:

1) A subgroup $\mathbf{T}$ of $(\mathbb{R}, +)$, called the **time set**, usually $\mathbb{R}$ or $\mathbb{Z}$

2) A non-empty set $\mathbf{X}$ called the **state space**

3) A non-empty set $\mathbf{U}$ called the **control value space**

4) a map $f : D_f \to \mathbf{X}$, called the **transition map** defined on some subset D_f of $\{(t,\, s,\, \mathbf{x},\, \omega) : s \leqq t \in T, \mathbf{x} \in \mathbf{X}, \omega \in \mathbf{U}[s,t)\}$ (Here, $\omega \in \mathbf{U}[s,t)$ indicates a mapping from the interval $[s,t) \subset \mathbf{T}$ to the state space $\mathbf{U}$.)

With Properties:

A) For all $x \in \mathbf{X}$ and $s \in \mathbf{T}$, there is $t > s$ and some $\omega \in \mathbf{U}[s,\, t)$ such that ω is **admissible** for x; that is, $(t,\, s,\, \mathbf{x},\, \omega)$ is in the domain of the transition map f. (nontriviality)

B) $\omega \in \mathbf{U}[s,m)$ is admissible for $\mathbf{x}$, then for all $t \in [s,m)$, the restriction ω_1 to the subinterval $[s,t)$ is also admissible for $\mathbf{x}$ and the restriction ω_2 to the subinterval $[t,m)$ is admissible for $f(t,s,x,\omega_1)$ (restriction)

C) Given $s < t < m \in \mathbf{T}$, if $\omega_1 \in \mathbf{U}[s,t)$ and $\omega_2 \in \mathbf{U}[t,m)$ and $f(t,s,\mathbf{x},\omega_1) = \mathbf{x}_1$ and $f(m,t,\mathbf{x}_1,\omega_2) = \mathbf{x}_2$, then ω, the control given by

$$\omega(\tau) = \begin{cases} \omega_1(\tau) & \text{for} \quad \tau \in (s,t] \\ \omega_2(\tau) & \text{for} \quad \tau \in (t,m], \end{cases}$$

is also admissible for $\mathbf{x}$ and $f(m,s,\mathbf{x},\omega) = \mathbf{x}_2$ (semigroup)

D) For each $s \in \mathbf{T}$. and each $x \in \mathbf{X}$, the empty sequence $\varnothing \in \mathbf{U}[s,s]$ is admissible for x and $f(s,s,x,\varnothing) = x$ (identity)

In a nutshell, what 1-4 mean is that a machine Σ is a a collection of functions over a set $\mathbf{T}$. The state space $\mathbf{X}$ tells us what state the machine is in. The elements of $\mathbf{X}$ are called **states**, or **configurations**. For the cart and double pendulum system, a state is defined by the positions and velocities of the cart and two pendulum bobs. The control value space $\mathbf{U}$ tells us what we are doing to the machine. In this case, we are applying a one-dimensional acceleration to the cart. The transition map tells us where we *steer* to with a set of control values $\omega(s,t]$ starting at state x and time s and ending at time t. In the case of the cart and double-pendulum system, we will not explicitly define a function $D_f \mapsto \mathbf{X}$. Instead, we will use a system of differential equations to describe the motion.

Properties A through D are straightforward, but I will not invoke them here because my systems do not have non-admissible controls. Systems, such as mine, that have no non-admissible controls, are called **complete systems**. Property A says that for every state at every time, there is at least one admissible control. Properties B and C say that admissible controls may be split up and concatenated. Property D defines the *empty sequence* control on a null time set. There are some models that I will use where there is no input acceleration. These are called **machines with no controls**. We characterize these machines by a control set $\mathbf{U}$ that contains only the null control.

Consider the system I just introduced with just a cart and an acceleration on the cart. The time set is continuous, so $\mathbf{T} = \mathbb{R}$. The configuration of the system is given by the displacement of the cart and the input is given by the acceleration; both of these range over a continuum. Therefore, $\mathbf{X} = \mathbf{U} = \mathbb{R}$. For the cart and single-pendulum system, we add a circle to the configuration space so that $\mathbf{X} = \mathbb{R} \times \mathbf{S}$ For the cart and double-pendulum system, we add yet another circle to the configurations space so that $\mathbf{X} = \mathbb{R} \times \mathbf{S}^2$.

Now, I will show how a completely different system fits into this framework. Imagine a 'black box' with a display that shows a number. Once every second, the black box adds up the digits in the number and adds the sum to the number on the display. There is a button on the black box. Each time the button is pressed, the most significant and least significant digits are switched. At $t = 0$, the display reads 1. Can we prevent the display from ever going over 100?

We can model the black box with $\mathbf{T} = \mathbb{Z}$, $\mathbf{X} = \mathbb{N}$, $\mathbf{U} = \textit{binary set}$. The set $\mathbf{T}$ can be modeled as the set of integers since we can model the black box as a machine who's state changes once per second. That the set $\mathbf{X} = \mathbb{N}$ follows from the fact that the state of the black box is entirely determined by the number on the display. That the set $\mathbf{U}$ is binary follows from the fact the we can do one of two things for either state: permute the digits, or not.

Definition 4.2. A **path** is a function $p : [s, t) \mapsto \mathbf{X}$ such that there is some control $\omega \in \mathbf{U}[s, t)$ that steers the system from state $p(s)$ to $p(t)$.

Definition 4.3. A **trajectory** is a path p from $p(s)$ to $p(t)$ and the control $\omega[s, t)$ which steers the system from $p(s)$ to $p(t)$.

There are several subsets of machines that are of interest to us. The 'black box' example is an example of what are known as **discrete-time machines**, characterized by $\mathbf{T} = \mathbb{Z}$. Machines that are modeled by continuous differential equations are known as **continuous-time machines**. Although there are additional criterion for this term to apply, I believe that they are beyond the scope of my research. It will suffice to say that if the state space and control space are Cartesian products of the reals and $\frac{d}{dt}\mathbf{x}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t))$, with f continuous, Σ is a continuous-time machine. The ticket to stabilization is going to be approximating the cart and double-pendulum model by a **linear continuous-time machine**. There is extensive theory as to how to stabilize systems where $\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$. The simplified system with only a cart is already a linear continuous-time system, but I will have to use a linearization (an approximation) about the up unstable equilibrium to use that theory for the cart and double-pendulum. [3]

Finally, beyond the scope of my research is an entirely different class of machines called **machines with outputs**. These are machines such that we cannot observe the configuration of the machine at any time. Rather, we observe the value of some function of the configuration, control, and time. This could be as simple as a projection, where we may not consider the value of certain parameters in the configuration space, a time translation, or round-off error. A system that models the price of a stock would be one such example.

5. Kinetic and Potential Energy

In order to calculate the equations of motion for the two systems, it will be necessary to determine the kinetic and potential energies of the systems. The total kinetic energy of the system is the sum of the kinetic energies of all the particles. The total potential energy of the system is the sum of the potential energies of all the particles. I introduce the following notation.

$$\begin{aligned}
hd(particle) &= \text{the horizontal displacement of a point particle from a reference point}\\
vd(particle) &= \text{the vertical displacement of a point particle from a reference point}\\
disp(particle) &= \text{the displacement of a point particle from a reference point in Euclidean space}\\
speed(particle) &= \text{the speed of a point particle from a reference point in Euclidean space}\\
mass(particle) &= \text{the mass of a point particle}
\end{aligned} \tag{5.1}$$

Remember that I am modeling the cart, and both pendulums as point masses to avoid calculating the kinetic energy due to rotational motion. The masses of the pendulums are concentrated at points on the pendulum bobs while the mass of the cart is concentrated at the pivot joint.

Kinetic and potential energy are given as

$$\begin{aligned}
KE(particle) &= \frac{1}{2} mass(particle)\, speed(particle)^2\\
PE(particle) &= mass(particle)\, vd(particle)\, g
\end{aligned} \tag{5.2}$$

The masses of the particles and acceleration due to gravity are held constant in my model. Since the cart is constrained to move only horizontally, $vd(cart) = 0$, $hd(cart) = x(t)$, and $disp(cart) = x(t)$. The first pendulum moves in a circle of radius l_p around the cart. Therefore, I calculate $vd(pendulum_1) = vd(cart) + l_p \cos(\theta(t))$ and $hd(pendulum_1) = hd(cart) + l_p \sin(\theta(t))$. In the cart and double-pendulum case, the second pendulum moves in a circle of radius l_q around the first pendulum bob the same way that the first pendulum rotates around the cart. Therefore, $vd(pendulum_2) = vd(pendulum_1) + l_q \cos(\phi(t))$ and $hd(pendulum_2) = hd(pendulum_1) + l_q \sin(\phi(t))$. When I plug values into these equations, I get

$$\begin{aligned}
KE(cart) &= \frac{1}{2} M\, (\frac{d}{dt}x(t))^2\\
KE(pendulum_1) &= \frac{1}{2} p \left(((\frac{d}{dt}x(t)) + l_p \cos(\theta(t))(\frac{d}{dt}\theta(t)))^2 + l_p^2 \sin(\theta(t))^2\, (\frac{d}{dt}\theta(t))^2 \right)\\
KE(pendulum_2) &= \frac{1}{2} q \left(((\frac{d}{dt}x(t)) + l_p \cos(\theta(t))\, (\frac{d}{dt}\theta(t)) + l_q \cos(\phi(t))\, (\frac{d}{dt}\phi(t))^2 \right.\\
&\quad \left. + (-l_p \sin(\theta(t))(\frac{d}{dt}\theta(t)) - l_q \sin(\phi(t))\, (\frac{d}{dt}\phi(t)))^2 \right)
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
PE(cart) &= 0\\
PE(pendulum_1) &= p\, l_p \cos(\theta(t))\, g\\
PE(pendulum_2) &= q\, (l_p \cos(\theta(t)) + l_q \cos(\phi(t)))\, g
\end{aligned} \tag{5.4}$$

Note that the energies of the cart and first pendulum are the same in both systems. The only difference between the cart and pendulum and the cart and double-pendulum system is the energy of the second pendulum and the variable $\phi(t)$.

6. The Calculus of Variations, The Forced Euler-Lagrange Equations, And The Lagrange-D'Alembert Principle

In describing the equations of motion of mechanical systems, I introduce the Euler-Lagrange Equations, a fundamental result from the calculus of variations. According to the Principle of Least Action, action is minimized over time by the motion of a system with no external forces. Action is given by the integral of the difference between the total kinetic and total potential energies in a system, which are described in terms of time, state, and velocity (the time derivative of state). In the case of the cart and pendulum, I write $\mathbf{q} = \left(x(t), \quad \theta(t) \right)$ and $\mathbf{v} = \left(\frac{d}{dt}x(t), \quad \frac{d}{dt}\theta(t) \right)$, for example.

$$Action = \left(\int_{t=t_0}^{t_f} (KE - PE)(t, \mathbf{q}, \mathbf{v}) \, dt \right) \tag{6.1}$$

Since the components of $\mathbf{v}$ are the time-derivatives of the components of $\mathbf{q}$, it will suffice to find a function $\mathbf{q}(t)$ that minimizes action. The calculus of variations shows us how to minimize this quantity. In ordinary calculus, we have techniques for minimizing a function of several variables. What the calculus of variations is all about is working with functions of a continuum of variables. Specifically, a main result of the calculus of variations is that a minimizer of the action satisfies the Euler-Lagrange Equations, given below. It is this result that allows me to describe the motion of mechanical systems. [4]

Here are the Euler-Lagrange Equations:

For i from 1 to $\dim(\mathbf{q})$,

$$-\frac{\partial}{\partial \mathbf{q}_i}(KE - PE)(t, \mathbf{q}, \mathbf{v}) + \frac{d}{dt}\frac{\partial}{\partial \mathbf{v}_i}(KE - PE)(t, \mathbf{q}, \mathbf{v}) = 0 \tag{6.2}$$

Next, I use the Lagrange-D'Alembert Principle to add the external force to the system. For the purpose of this paper, we may describe this principle by

$$(mass)(acceleration) - (potential\, force) = (External\, Forces) \tag{6.3}$$

The left-hand side of the Euler-Lagrange equations goes on the left-hand side of this equation. On the right-hand side, we have $M\,u(t)$, mass times acceleration of the cart in the first line, and 0 everywhere else. These are called the Forced Euler-Lagrange Equations [4]. A more extensive explanation of the Forced Euler-Lagrange Equations can be found in [4].

For i from 1 to $\dim(\mathbf{q})$,

$$-\frac{\partial}{\partial \mathbf{q}_i}(KE - PE)(t, \mathbf{q}, \mathbf{v}) + \frac{d}{dt}\frac{\partial}{\partial \mathbf{v}_i}(KE - PE)(t, \mathbf{q}, \mathbf{v}) = F_i \quad (6.4)$$

It is critical to understand that the left-hand side is evaluated as follows: First, compute KE and PE in terms of $\mathbf{q}$ and $\mathbf{v}$. Then, differentiate each component with respect to $\mathbf{q}_i$ and $\mathbf{v}_i$. Next, take the time derivative of the $\mathbf{v}_i$ derivative, subtract from the $\mathbf{q}_i$ derivative.

7. The Equations of Motion For The Cart and Single-Pendulum And The Cart and Double Pendulum

Now that I have described the Forced Euler-Lagrange Equations, I apply them to the cart and pendulum system, but first, I define constants and variables that appear in my equations:

Here is the quantity in the integrand of the action; the difference between kinetic and potential energy in the system. In this equation, I write v_x and v_θ, respectively, for $\frac{dx}{dt}$ and $\frac{d\theta}{dt}$.

$$KE - PE = \frac{M+p}{2}\, v_x^2 + p\, l_p\, \cos(\theta) v_\theta v_x + \frac{1}{2}\, p\, l_p^2\, v_\theta^2 - g\, p\, l_p\, \cos(\theta) \quad (7.1)$$

We need to get the derivatives of this quantity with respect to $\mathbf{q}$ and $\mathbf{v}$.

$$\begin{aligned} &dim(\mathbf{q}) = 2 \\ &\frac{\partial}{\partial \mathbf{q}_1}(KE - PE) = \frac{\partial}{\partial x}(KE - PE) = 0 \\ &\frac{\partial}{\partial \mathbf{q}_2}(KE - PE) = \frac{\partial}{\partial \theta}(KE - PE) = -p\, l_p\, \sin(\theta) v_\theta\, v_x - g\, p\, l_p\, \sin(\theta) \\ &\frac{\partial}{\partial \mathbf{v}_1}(KE - PE) = \frac{\partial}{\partial v_x}(KE - PE) = l_p\, M\, v_x + p\, v_x p\, l_p\, \cos(\theta)\, v_\theta \\ &\frac{\partial}{\partial \mathbf{v}_2}(KE - PE) = \frac{\partial}{\partial v_\theta}(KE - PE) = p\, l_p\, \cos(\theta)\, v_x + p\, l_p^2\, v_\theta \end{aligned} \quad (7.2)$$

When we plug these derivatives and the input force into the Forced Euler-Lagrange equations, the result is the equations of motion for the cart and single-pendulum. Remember, force equals mass times acceleration. Here are the equations of motion

$$\begin{aligned} (M+p)\frac{d^2}{dt^2}x(t) - p\, l_p\, \sin(\theta(t))(\frac{d}{dt}\theta(t))^2 + p\, l_p\, \cos(\theta(t))\, \frac{d^2}{dt^2}\theta(t) = M\, u(t) \\ p\, l_p\, \cos(\theta(t))\frac{d^2}{dt^2}x(t) + p\, l_p^2\, \frac{d^2}{dt^2}\theta(t) - g\, p\, l_p\, \sin(\theta(t)) = 0 \end{aligned} \quad (7.3)$$

I then isolate the accelerations:

$$\frac{d^2}{dt^2}x(t) = \frac{g\,\cos(\theta(t))p\,\sin(\theta(t)) + p\,l_p\,\sin(\theta(t))(\frac{d}{dt}\theta(t))^2 + M\,u(t)}{M + p - p\,\cos(\theta(t))^2}$$

$$\frac{d^2}{dt^2}\theta(t) = \frac{p\,\cos(\theta(t))\,M\,u(t) + p^2\,\cos(\theta(t))\,l_p\,\sin(\theta(t))\,(\frac{d}{dt}\theta(t))^2 - p\,g\,\sin(\theta(t))\,M - p\,g\,\sin(\theta(t))\,p}{M + p - p\,\cos(\theta(t))^2)\,p\,l_p} \tag{7.4}$$

For the cart and double-pendulum system, I apply the exact same formulation for the equations of motion but with different values for kinetic and potential energy. Specifically, we just add the kinetic and potential energies of the second pendulum bob and we get a three dimensional system.

$$\begin{aligned}
\frac{d^2}{dt^2}x(t) &= -\frac{(2\,l_p\,p^2\,\sin(\theta(t)) + 2\,l_p\,p\,q\,\sin(\theta(t)))\,(\frac{d}{dt}\theta(t))^2}{D(t,\theta(t),\phi(t))} \\
&- \frac{(p\,q\,l_q\,\sin(2\,\theta(t) - \phi(t)) + p\,q\,l_q\,\sin(\phi(t)))\,(\frac{d}{dt}\phi(t))^2}{D(t,\theta(t),\phi(t))} \\
&- \frac{(-p^2\,\sin(2\,\theta(t)) - p\,q\,\sin(2\,\theta(t)))\,g}{D(t,\theta(t),\phi(t))} \\
&+ \frac{(2\,M\,p + M\,q - M\,q\,\cos(2\,\theta(t) - 2\,\phi(t)))\,u(t)}{D(t,\theta(t),\phi(t))} \\
\frac{d^2}{dt^2}\theta(t) &= \frac{(l_p\,p^2\,\sin(2\,\theta(t)) + l_p\,p\,q\,\sin(2\,\theta(t)) + l_q\,M\,q\,\sin(2\,\theta(t) - 2\,\phi(t)))\,(\frac{d}{dt}\theta(t))^2}{l_p\,(D(t,\theta(t),\phi(t)))} \\
&- \frac{(p\,q\,l_q\,\sin(\theta(t) + \phi(t)) + p\,q\,l_q\,\sin(\theta(t) - \phi(t)) + 2\,M\,q\,l_q\,\sin(\theta(t) - \phi(t)))\,(\frac{d}{dt}\phi(t))^2}{l_p\,(D(t,\theta(t),\phi(t)))} \\
&- \frac{M\,p\,(-2\,\sin(\theta(t)) - M\,q\,\sin(\theta(t)) - 2\,p^2\,\sin(\theta(t)) - 2\,p\,q\,\sin(\theta(t)) - M\,q\,\sin(\theta(t)) - 2\,\phi(t))\,g}{l_p\,(D(t,\theta(t),\phi(t)))} \\
&+ \frac{(2\,M\,p\,\cos(\theta(t)) + M\,q\,\cos(\theta(t)) - M\,q\,\cos(\theta(t) - \phi(t)))\,u(t)}{l_p\,(D(t,\theta(t),\phi(t)))} \\
\frac{d^2}{dt^2}\phi(t) &= -\frac{(-2\,M\,p\,l_p\,\sin(\theta(t) - \phi(t)) - 2\,M\,q\,l_q\,\sin(\theta(t) - \phi(t)))\,(\frac{d}{dt}\theta(t))^2}{l_q\,(D(t,\theta(t),\phi(t)))} \\
&+ \frac{M\,q\,l_q\,\sin(2\,\theta(t) - 2\,\phi(t))\,(\frac{d}{dt}\phi(t))^2}{l_q\,(D(t,\theta(t),\phi(t)))} \\
&- \frac{(M\,p\,\sin(2\,\theta(t) - \phi(t)) - M\,p\,\sin(\phi(t)) + M\,q\,\sin(2\,\theta(t) - \phi(t)) - M\,q\,\sin(\phi(t)))\,g}{l_q\,(D(t,\theta(t),\phi(t)))} \\
&+ \frac{(M\,p\,\cos(\phi(t)) - M\,p\,\cos(2\,\theta(t) - \phi(t)) + M\,q\,\cos(\phi(t)) - M\,q\,\cos(2\,\theta(t) - \phi(t)))\,u(t)}{l_q\,(D(t,\theta(t),\phi(t)))} \\
D(t,\theta(t),\phi(t)) &= p^2 + 2\,M\,p + M\,q + p\,q - M\,q\,\cos(2\,\theta(t) - 2\,\phi(t)) - p\,q\,\cos(2\,\theta(t)) - p^2\,\cos(2\,\theta(t))
\end{aligned} \tag{7.5}$$

8. Sunglasses

When you encounter these equations for the first time, as I did, you will no doubt be overwhelmed by their complexity. Many researchers who have worked with the cart and double-pendulum system have called the system's behavior "chaotic" owing to the fact that these equations seem so opaque.

Here, I provide some vector-field plots which have helped me understand the motion of the system. I call these graphs my "sunglasses" because they allow me to see what the eye normally cannot look at.

First, notice that the equations take the form

$$\frac{d^2}{dt^2}x(t) = A_1(\theta(t),\phi(t))\,g + A_2(\theta(t),\phi(t))\,u(t) + A_3(\theta(t),\phi(t))\,(\frac{d}{dt}\theta(t))^2 + A_4(\theta(t),\phi(t))\,(\frac{d}{dt}\phi(t))^2 \tag{8.1}$$

$$\frac{d^2}{dt^2}\theta(t) = B_1(\theta(t),\phi(t))\,g + B_2(\theta(t),\phi(t))\,u(t) + B_3(\theta(t),\phi(t))\,(\frac{d}{dt}\theta(t))^2 + B_4(\theta(t),\phi(t))\,(\frac{d}{dt}\phi(t))^2 \tag{8.2}$$

$$\frac{d^2}{dt^2}\phi(t) = C_1(\theta(t),\phi(t))\,g + C_2(\theta(t),\phi(t))\,u(t) + C_3(\theta(t),\phi(t))\,(\frac{d}{dt}\theta(t))^2 + C_4(\theta(t),\phi(t))\,(\frac{d}{dt}\phi(t))^2 \tag{8.3}$$

Thus, we can compute separately the effects of (1) Gravity (2) The External Force and (3) The pendulums torquing each other.

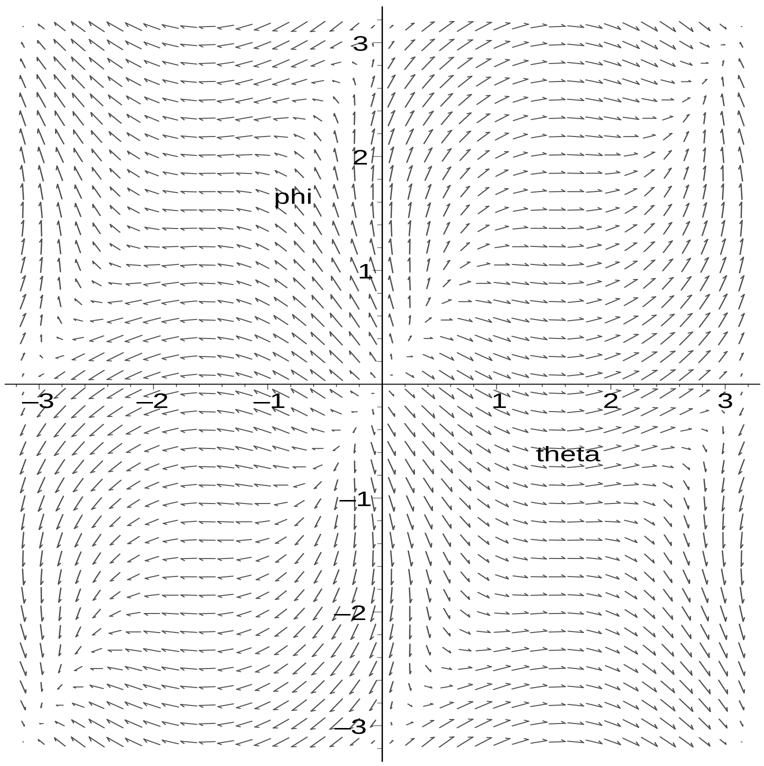
I will assume that a *reasonable* system will have a cart that is much heavier than the pendulum bobs. I consider it unreasonable to to expect these equations to model the behavior of a cart and double-pendulum with pendulum bobs heavier than the cart because if we build such a system, one end of the cart will likely lift off the track as it is torqued by the pendulum arms, violating the model's condition that the cart moves exclusively in a straight line.

Notice that the right-hand side of the equations of motion do not contain the position, velocity, or acceleration of the cart. They contain functions of $\theta(t)$, $\phi(t)$, $u(t)$, and g. The bottom line is that to understand the behavior of the two pendulum bobs, it will suffice to understand the coefficients of g, $u(t)$, $(\frac{d}{dt}\theta(t))^2$, and $(\frac{d}{dt}\phi(t))^2$.

For now, let us consider only the acceleration of the two pendulum bobs. It will be desirable to control the motion of the cart as well, but I'll come back to that later.

Here are the values I use for the constants in the system here:

$M = 5$, $p = 1$, $q = 1$, $l_p = 1$, $l_q = 1$



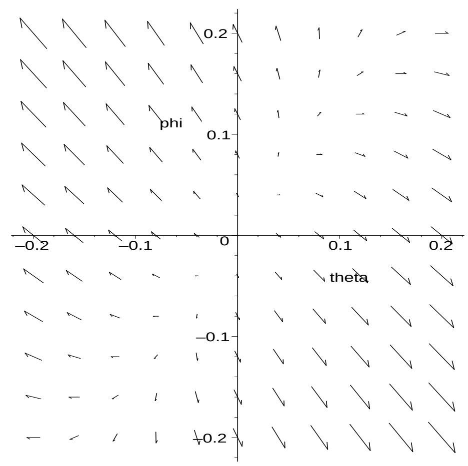


FIGURE B.1a *FIGURE B.1b*

Figure B.1a shows, on a (θ, ϕ) axis, $B_1(\theta(t), \phi(t))$, the coefficient of g in the acceleration of the first pendulum bob, and $C_1(\theta(t), \phi(t))$, the coefficient of g in the acceleration of the second pendulum bob from (8.1). Examine Figure B.1b, a zoomed in view of the graph close to the point $(0, 0)$. Recall that this point represents the up unstable equilibrium. At the majority of the points in a small neighborhood of $(0, 0)$ we see that gravity accelerates both pendulum bobs away from $(0, 0)$ as we expect. However, there are two regions of points where gravity accelerates the first pendulum bob away from the unstable equilibrium and accelerates the second pendulum bob toward the unstable equilibrium. This is an important observation which will appear again in the *Simulation* section. Specifically, for the values of the constants I have used here, the first region is approximately the area below $\theta = \phi$ in the first quadrant on the graph. The second region is approximately the area above $\theta = \phi$ in the third quadrant.

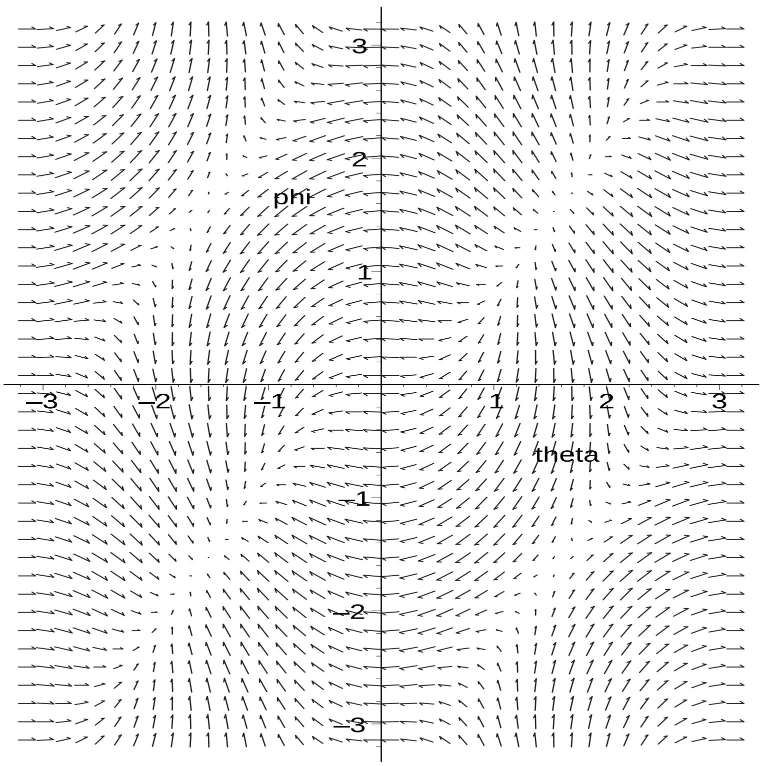


FIGURE B.2a

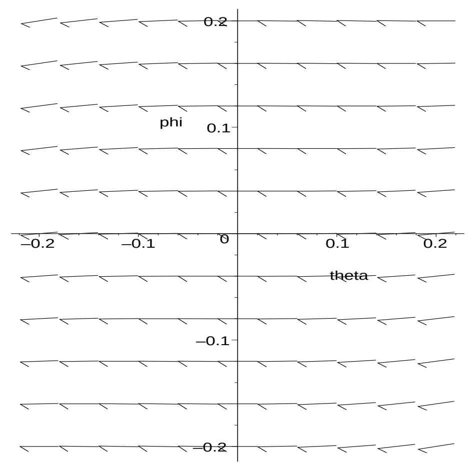


FIGURE B.2b

Figure B.2a shows, on a (θ, ϕ) axis, $B_2(\theta(t), \phi(t))$, the coefficient of $u(t)$ in the acceleration of the first pendulum bob, and $C_2(\theta(t), \phi(t))$, the coefficient of $u(t)$ in the acceleration of the second pendulum bob from (8.1). Examine Figure B.2b, a zoomed in view of the graph close to the point $(0, 0)$. Just as Figure B.1b showed us how the two pendulum bobs accelerate due to gravity near the unstable equilibrium, Figure B.2b shows how the two pendulum bobs accelerate due to the input force, only this time, the data is much more straight forward! We push the cart in one direction with both pendulums in the up position. According to Newton's Second Law of Motion, there is an equal and opposite reaction. At the up unstable equilibrium, that equal and opposite reaction is a torque in the opposite direction of the first pendulum. At this moment, there is no action on the second pendulum bob. Since I will need to stay in a neighborhood of the unstable equilibrium, I will approximate $B_2(\theta(t), \phi(t))$ and $C_2(\theta(t), \phi(t))$ by their values at this point. We shall see that for stabilization against small disturbances, a model using this approximation is sufficient

$$\begin{aligned} B_2(\theta(t), \phi(t)) &\approx B_2(0,0) = \frac{-1}{l_p} \\ C_2(\theta(t), \phi(t)) &\approx C_2(0,0) = 0 \end{aligned} \tag{8.4}$$

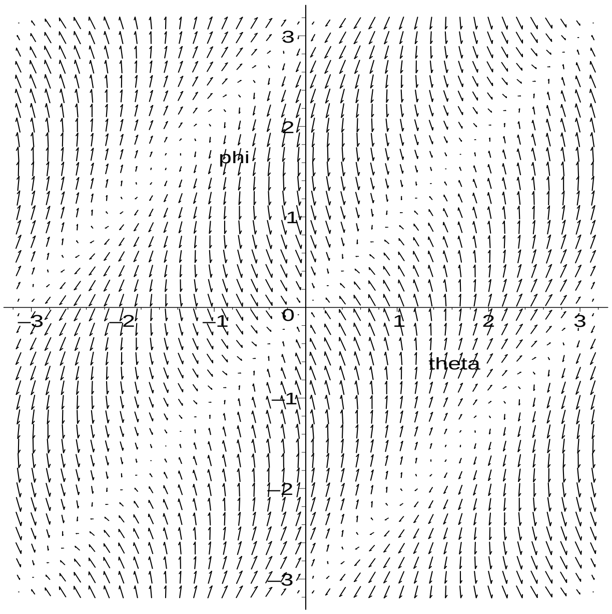


FIGURE B.3

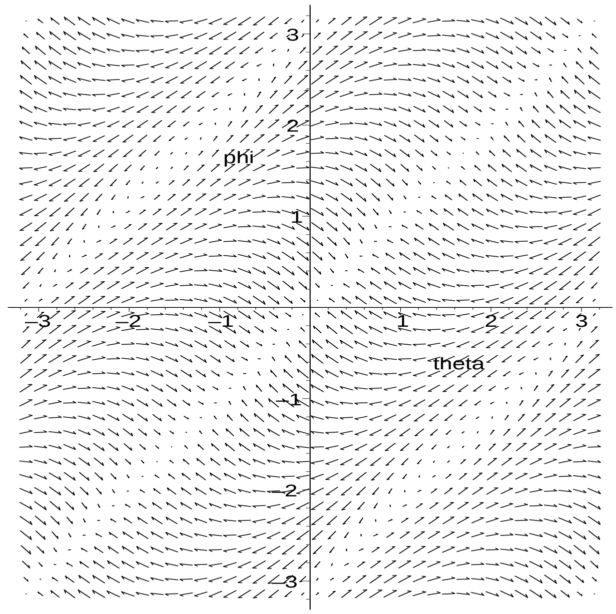


FIGURE B.4

Figure B.3 shows $B_3(\theta,\phi)$ and $C_3(\theta,\phi)$, the coefficients of $(\frac{d}{dt}\theta(t))^2$ in the accelerations of the two pendulum bobs from (8.1). Figure B.4 shows $B_4(\theta,\phi)$ and $C_4(\theta,\phi)$, the coefficients of $(\frac{d}{dt}\phi(t))^2$. As you can see, these graphs show that as the pendulum bobs are accelerated, they tend towards a configuration where the two pendulum arms form a straight line. However, when I linearize, I will be approximating all these quantities as 0. These calculations will be shown in the following section.

9. Linearization of the Equations of Motion

I now present the linearization of the equations of motion of the cart and single-pendulum system and the cart and double-pendulum system. We will see that for stabilization against small disturbances, such a model will suffice. Let's begin with the cart and single-pendulum system.

Definition 9.1. Let Σ be a continuous-time machine governed by the equation $\frac{d}{dt}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$. The **linearization** of Σ about $\mathbf{x} = 0, \mathbf{u} = 0$ is the linear continuous-time machine governed by the equation $\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$.

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}}\right|_{(\mathbf{x},\mathbf{u})=(0,0)} \qquad \mathbf{B} = \left.\frac{\partial \mathbf{f}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}\right|_{(\mathbf{x},\mathbf{u})=(0,0)} \tag{9.1}$$

To compute the linearizations of the cart and single-pendulum system, I must transform my equations of motion into a system of first-order differential equations. An easy way to do this is to write new variables for the velocities and relate them to the configuration by differential equations. For this system, I write

$$
\begin{aligned}
\frac{d}{dt}v_x(t) &= \frac{g\,\cos(\theta(t))p\,\sin(\theta(t)) + p\,l_p\,\sin(\theta(t))v_\theta^2 + M\,u(t)}{M + p - p\,\cos(\theta(t))^2} \\
\frac{d}{dt}v_\theta(t) &= \frac{p\,\cos(\theta(t))\,M\,u(t) + p^2\,\cos(\theta(t))\,l_p\,\sin(\theta(t))\,v_\theta^2 - p\,g\,\sin(\theta(t))\,M - p\,g\,\sin(\theta(t))\,p}{M + p - p\,\cos(\theta(t))^2)\,p\,l_p} \\
\frac{d}{dt}x(t) &= v_x(t) \\
\frac{d}{dt}\theta(t) &= v_\theta(t)
\end{aligned}
\tag{9.2}
$$

Not only have I transformed the equations of motion from a system of second-order differential equations to a system of first order differential equations, I have also introduced the velocities as dimensions in the state space. For these equations and their linearization, $\mathbf{x}(t) = \begin{pmatrix} v_x(t), & v_\theta(t), & x(t), & \theta(t) \end{pmatrix}^T$. I now compute the linearization by 9.1. I obtain

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$

where

$$
\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & \frac{-m\,g}{M} \\ 0 & 0 & 0 & \frac{-m\,g\,M - m\,g\,p}{M\,l_p\,p} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\qquad
\mathbf{B} = \begin{pmatrix} 1 \\ \frac{-1}{l_p} \\ 0 \\ 0 \end{pmatrix}
$$

For the cart and double-pendulum system, I apply a transformation to the equations as before my system of three second-order differential equations becomes a system of six first-order differential equations. $\mathbf{x}(t) = \begin{pmatrix} v_x(t), & v_\theta(t), & v_\phi(t), & x(t), & \theta(t), & \phi(t) \end{pmatrix}^T$

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$

where

$$
\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{-q\,g\,p - p^2\,g}{p\,M} & 0 \\ 0 & 0 & 0 & 0 & \frac{-q\,g\,M - q\,g\,p - p\,g\,M - p^2\,g}{l_p\,p\,M} & \frac{q\,g}{p\,l_p} \\ 0 & 0 & 0 & 0 & \frac{-q\,g - g\,p}{p\,l_q} & \frac{g\,p + g\,q}{p\,l_q} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}
\qquad
\mathbf{B} = \begin{pmatrix} 1 \\ \frac{-1}{l_p} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\tag{9.3}
$$

Notice again the lack of influence of the position and velocity of the cart on the acceleration of the two pendulum bobs.

10. Linear State Feedback

Before I proceed to describe the controller that I use, I would like to briefly introduce a theorem from linear control theory regarding when a linear controller exists. The theorem shows that a controller exists for the linear system regardless of the constants that I use for the mass, length, gravitational force, and so on.

Lemma 10.1. *(Linear Reachability) The linear system* $\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$ *can be steered between arbitrary points by a linear controller if and only if the matrix* $\begin{pmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\,\mathbf{B} & \dots & \mathbf{A}^{dim(x)}\mathbf{B}\end{pmatrix}$ *is of maximal rank. [6]*

For both the cart and single-pendulum and cart and double-pendulum systems, we get square reachability matrices with non-zero determinants for all non-zero values for the constants in the system. The proof of this theorem is rather complicated. I simply include it here as a remark to show that the choice of physical constants is not critical for these methods to apply.

With the equations of motion linearized about the up unstable equilibrium, I now apply techniques in linear control theory to stabilize the cart and double-pendulum system against small disturbances.

A very straight-forward method for doing this that I have come across is linearized state feedback. The point I wish to stabilize about is zero in my coordinates. Make the substitution $\mathbf{u}(t) = \mathbf{F}\,\mathbf{x}(t)$. Then,

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$

becomes

$$\frac{d}{dt}\mathbf{x}(t) = (\mathbf{A} + \mathbf{B}\,\mathbf{F})\mathbf{x}(t) \tag{10.1}$$

The solution to this differential equation is

$$\begin{aligned} \mathbf{x}(t) &= e^{t\,\mathbf{M}}\mathbf{x}(0) \\ \mathbf{M} &= \mathbf{A} + \mathbf{B}\,\mathbf{F} \end{aligned} \tag{10.2}$$

Definition 10.2. A matrix $\mathbf{A}$ is called a **Hurwitz matrix** if its eigenvalues all have negative real part.

Theorem 10.3. *(Jordan-Chevalley Decomposition Theorem) If* $\mathbf{A}$ *is a square matrix, then* $\mathbf{A} = \mathbf{S} + \mathbf{N}$ *where* $\mathbf{S}$ *is diagonalizable over* $\mathbb{C}$ *and has the same eigenvalues as* $\mathbf{M}$*, and* $\mathbf{N}$ *is nilpotent;* $(\exists\, n \ni \mathbf{N}^n = 0)$ *and* $\mathbf{S}\,\mathbf{N} = \mathbf{N}\,\mathbf{S}$.

Remark 10.4. This follows from the Jordan Canonical Form representation of $\mathbf{A}$.

Theorem 10.5. *(Linear Stability Theorem) If the matrix $\mathbf{M}$ is a Hurwitz matrix, then the system $\frac{d}{dt}\mathbf{x}(t) = \mathbf{M}\,\mathbf{x}(t)$, whose solution is $\mathbf{x}(t) = e^{t\,\mathbf{M}}\mathbf{x}(0)$, stabilizes to 0.*

Proof. In the one-dimensional case, it will suffice for $\mathbf{M}$'s single entry to have negative real part. For the higher dimensional case, a sufficient condition for $\mathbf{x}(t)$ to go to zero is for the eigenvalues of $\mathbf{M}$ to have negative real part. To see why, consider the MacLauren expansion of the matrix exponential:

$$e^{t\,\mathbf{M}} = I + t\,\mathbf{M} + \frac{t^2\,\mathbf{M}^2}{2!} + \frac{t^3\,\mathbf{M}^3}{3!} + \frac{t^4\,\mathbf{M}^4}{4!} + \frac{t^5\,\mathbf{M}^5}{5!} + \frac{t^6\,\mathbf{M}^6}{6!} + \frac{t^7\,\mathbf{M}^7}{7!} + \dots \tag{10.3}$$

First, consider the case that $\mathbf{M}$ is a diagonal matrix. In this case, the vector equation decouples into $dim(\mathbf{M})$ scalar equations so that for each n from 1 to $dim(\mathbf{M})$,

$$\frac{d}{dt}\mathbf{x}_n(t) = \mathbf{M}_{n,n}\,\mathbf{x}_n(t) \tag{10.4}$$

If $\mathbf{M}$ is diagonalizable, $\mathbf{M} = \mathbf{P}^{-1}\,\mathbf{D}\,\mathbf{P}$, where $\mathbf{P}$ is invertible and $\mathbf{D} = diag(e_1, e_2, \dots, e_{dim(\mathbf{M})})$ is the diagonal matrix of the eigenvalues of $\mathbf{M}$. It follows from $(\mathbf{PDP^{-1}})^2 = \mathbf{PD}^2\mathbf{P}^{-1}$ that

$$\begin{aligned} e^{t\,\mathbf{M}} = \mathbf{P}^{-1}\left(I + t\,\mathbf{D} + \frac{t^2\,\mathbf{D}^2}{2!} + \frac{t^3\,\mathbf{D}^3}{3!} + \frac{t^4\,\mathbf{D}^4}{4!} + \frac{t^5\,\mathbf{D}^5}{5!} + \frac{t^6\,\mathbf{D}^6}{6!} + \frac{t^7\,\mathbf{D}^7}{7!} + \dots\right)\mathbf{P} \\ = \mathbf{P}e^{t\,\mathbf{D}}\mathbf{P}^{-1} \end{aligned} \tag{10.5}$$

We can see from (10.5) that $e^{t\,\mathbf{M}}$ goes to 0 in this case as well. Finally, we are left with the case where $\mathbf{M}$ is not diagonalizable. Consider the Jordan-Chevelley decomposition of the matrix $\mathbf{M}$. By Lemma 10.3, $\mathbf{M} = \mathbf{S} + \mathbf{N}$ where $\mathbf{S}$ is diagonalizable, $\mathbf{N}$ is nilpotent, and $\mathbf{N}\,\mathbf{S} = \mathbf{S}\,\mathbf{N}$. Since the matrix multiplication commutes, it follows that $e^{t\,\mathbf{M}} = e^{t\,\mathbf{S}}\,e^{t\,\mathbf{N}}$.

The exponential of the diagonalizable matrix goes to 0 as before. It is possible that the exponential of the nilpotent matrix $e^{t\,\mathbf{N}}$ does not go to 0. It follows from the definition of nilpotency that the MacLauren expansion of this expression has a finite number of terms. Therefore, if $e^{t\,\mathbf{N}}$ does not go to zero, we can be assured that $e^{t\,\mathbf{N}}$ diverges in polynomial time at the fastest, so it will be overtaken by $e^{t\,\mathbf{S}}$. Therefore, $e^{\mathbf{M}} \to 0$.

□

It can now be seen that $\mathbf{A} + \mathbf{B}\,\mathbf{F}$ being Hurwitz is a sufficient condition to imply $\mathbf{x} \to 0$. Fortunately, when I did the math, I discovered that I may choose $\mathbf{F}$ to get whatever resulting real eigenvalues I desire for the matrix $\mathbf{A} + \mathbf{B}\,\mathbf{F}$. In most of my calculations, I have chosen values of $\mathbf{F}$ that yield eigenvalues of -1.

11. Lyapunov Stability

I use the Lyapunov Stability Theorem to show that the control $\mathbf{u}(t) = \mathbf{F}\,\mathbf{x}(t)$, with $\mathbf{A} + \mathbf{B}\,\mathbf{F}$ Hurwitz, stabilizes the cart and single-pendulum and cart and double-pendulum systems. As a matter of fact, what I present here is a theorem that can be applied to any time-invariant, continuous time system. I will state, but not prove a simplified version of the Lyapunov Stability Theorem. The proof is readily available in control theory literature [2]. In this section, think of the control as fixed at $\mathbf{u}(t) = \mathbf{F}\,\mathbf{x}(t)$ so that we get a new system with no controls. I call the new system Σ_{cl} and it's equations of motion $\mathbf{f}_{cl} = \frac{d}{dt}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{F}\,\mathbf{x}(t))$.

Definition 11.1. A **local Lyapunov function for the system** Σ_{cl} with no controls (relative to the equilibrium state 0) is a continuous function $V : \mathbf{X} \mapsto \mathbb{R}$ for which there is some neighborhood $\mathbf{O}$ of 0 such that the following properties hold:

(1) $\{\mathbf{x} \in \mathbf{X} \,|\, V(\mathbf{x}) \leqq \epsilon\}$ is a compact subset of $\mathbf{O}$ for each $\epsilon > 0$ small enough.

(2) $V(0) = 0$ and $V(x) > 0$ for each $x \in \mathbf{O}$, $\mathbf{x} \neq 0$

(3) For each $\mathbf{x} \neq 0 \in \mathbf{O}$, $\exists\, \sigma \in \mathbf{T}$, $\sigma > 0$, such that for the path $p(\mathbf{x})$ corresponding to this initial state, $V(p(t)) \leqq V(\mathbf{x})$ for all $t \in (0, \sigma]$ and $V(p(\sigma)) < V(\mathbf{x})$.

Lemma 11.2. *If Σ_{cl} is a continuous-time system with no controls and $V : \mathbf{X} \to \mathbb{R}$ is a continuous function with $\mathbf{O} \subseteq \mathbf{X}$ an open subset upon which V is continuously differentiable and properties (1) and (2) of a Lyapunov function hold, then the condition $\nabla V(\mathbf{x}) \cdot \mathbf{f}_{cl}(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathbf{O}$, $\mathbf{x} \neq 0$ is a sufficient condition for V to be a Lyapunov function.*

Remark 11.3. This follows from the fact that $\frac{d}{dt}V(p(t)) = \nabla V(\mathbf{x}) \cdot \mathbf{f}_{cl}(\mathbf{x})$.

Theorem 11.4. *If there exists a local Lyapunov function V for Σ_{cl}, a machine with no controls, then Σ is **locally asymptotically stable**. That is, if $\mathbf{x}(t)$ is sufficiently close to 0 for some t, then $\lim_{t \to \infty} \mathbf{x}(t) = 0$.*

Remark 11.5. The formal proof of this theorem is long and drawn out to cover every possible case. Recall that this theorem could describe a completely different machine from what we're thinking about, such as a discrete time system, for example. However, in the case of a continuous-time system, simply think of V as an energy function that gets smaller as $\mathbf{x}$ approaches 0 with $V = 0$ if and only if $\mathbf{x} = 0$. Therefore, when we show that V decreases along a path that the cart and double-pendulum system takes, we show that the cart and double-pendulum system approaches the up unstable equilibrium.

Lemma 11.6. *(Existence of Lyapunov Function for Stable Linear Systems) If $\mathbf{A}$ is a Hurwitz matrix, then there is a unique, positive definite solution to $\mathbf{A}^T\,\mathbf{P} + \mathbf{P}\,\mathbf{A} = -I$. Furthermore, $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\,\mathbf{x}$ is a Lyapunov function for a time-invariant linear system with no controls governed by the equation $\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\,\mathbf{x}(t)$.*

Remark 11.7. An important fact to take note of here is the fact that for a Lyapanov function determined in this manner, $\nabla V \cdot \frac{d}{dt}\mathbf{x}(t) = \frac{d}{dt}V = (\frac{d}{dt}\mathbf{x}(t))^T\mathbf{P}x + \mathbf{x}^T\mathbf{P}\frac{d}{dt}\mathbf{x}(t)$. Then make the substitutions $\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}$ and $\mathbf{P}\,\mathbf{A} = -I - \mathbf{A}^T\,\mathbf{P}$ to obtain $\frac{d}{dt}V(\mathbf{x}(t)) = -||\mathbf{x}(t)||^2 \leq 0$. I use this result to prove 11.8.

I now know that there is a Lyapunov function for the linearized system. I now proceed to prove that the Lyapunov function for the linear system is a local Lyapunov function for the non-linear system in a suitable neighborhood of 0.

Theorem 11.8. *(Linear Stabilization Principle) Let Σ be a time-invariant, continuous time system with*

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \tag{11.1}$$

where $\mathbf{f}(0,0) = 0$. If Σ_{lin}, the linearization around 0 of the system Σ, is stabilized around 0 asymptotically by the control $\mathbf{u}(t) = \mathbf{F}\,\mathbf{x}(t)$, then the system Σ is stabilized by the same control in a neighborhood of 0.

Proof. Substituting $\mathbf{u}(t) = \mathbf{F}\,\mathbf{x}(t)$ into

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \qquad\qquad \frac{d}{dt}\mathbf{x}(t) \approx \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,u(t)$$

yields

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{f}_{cl}(\mathbf{x}(t)) \qquad\qquad \frac{d}{dt}\mathbf{x}(t) \approx (\mathbf{A} + \mathbf{B}\,\mathbf{F})\,\mathbf{x}(t)$$

Next let

$$\mathbf{g}(\mathbf{x}, \mathbf{u}) = \mathbf{f} - (\mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)) \qquad\qquad \mathbf{h}(\mathbf{x}) = \mathbf{f}_{cl}(\mathbf{x}(t)) - (\mathbf{A} + \mathbf{B}\,\mathbf{F})\,\mathbf{x}(t)$$

It follows from the definition of linearization that

$$\lim_{||(\mathbf{x},\mathbf{u})||\to 0} \frac{||\mathbf{g}(\mathbf{x}, \mathbf{u})||}{||(\mathbf{x}, \mathbf{u})||} = 0 \qquad\qquad \lim_{||\mathbf{x}||\to 0} \frac{||\mathbf{h}(\mathbf{x})||}{||\mathbf{x}||} = 0. \tag{11.2}$$

Now, let V be a Lyapunov function for the linearized system $\frac{d}{dt}\mathbf{x}(t) = (\mathbf{A} + \mathbf{B}\,\mathbf{F})\mathbf{x}(t)$ as determined by the criterion in 11.6. It will suffice to show that $\nabla V \cdot \mathbf{f}_{cl} < 0$ in a neighborhood of 0. In other words, $\lim_{x\to 0} \nabla V \cdot \mathbf{f}_{cl} < 0$ is a sufficient condition.

$$\begin{aligned}\nabla V \cdot \mathbf{f}_{cl}(\mathbf{x}) = \nabla V \cdot ((\mathbf{A} + \mathbf{B}\,\mathbf{F})\mathbf{x}(t) + \mathbf{h}(\mathbf{x})) = \\ -||\mathbf{x}||^2 + 2\mathbf{x}^T\,\mathbf{P}\,\mathbf{h}(\mathbf{x})) = -||\mathbf{x}||^2\left(1 - \frac{2\mathbf{x}^T\mathbf{P}\,\mathbf{h}(\mathbf{x})}{||\mathbf{x}||^2}\right) < \\ -||\mathbf{x}||^2\left(1 - \frac{C\,||\mathbf{h}(\mathbf{x})||}{||\mathbf{x}||}\right) for\,some\,constant\,C.\end{aligned} \tag{11.3}$$

For the second equality, use the chain rule on the Lyapunov function V for the linear system whose time derivative is $(\mathbf{A} + \mathbf{B}\,\mathbf{F})\,\mathbf{x}(t)$.

In the last product, the factor $-||\mathbf{x}||^2$ is negative. The second factor goes to -1 since $\lim_{||x||\to 0} \frac{||h(x)||}{||x||} = 0$. Therefore, I establish that there is such a neighborhood in which the linear state feedback controller is guaranteed to stabilize the original, non-linear system. □

In my calculations, I estimate that the neighborhood in which the linear feedback controller is absolutely guaranteed by this theorem to stabilize a system modeled by the equations I use here is extremely small, I have found that there is a much larger neighborhood in which the system is most likely stabilized.

12. Simulation of the Motion of the Cart and Double-Pendulum

Here, I will give an example of a disturbance of the cart and double-pendulum system and show how my feedback controller stabilizes it. Suppose that the system has been slightly disturbed from the up-unstable equilibrium and one pendulum bob begins to fall one way and the other falls the other way. Let's say at time $t = 0$, we observe that the system is in the following configuration: $v_x(0) = 0, x(0) = 0, v_\theta(0) = .02, \theta(0) = .01, v_\phi(0) = -.02, \phi(0) = -.01$. Furthermore, let's assume as before that $M = 5$, $p = 1$, $q = 1$, $l_p = 1$, $l_q = 1$, and $g = 10$.

With the values for $\mathbf{A}$ and $\mathbf{B}$ that I introduced for the cart and double-pendulum system, I now use a controller of the form $\mathbf{u}(t) = \mathbf{F}\,\mathbf{x}(t)$ to obtain the system $\frac{d}{dt}\mathbf{x}(t) = (\mathbf{A} + \mathbf{B}\,\mathbf{F})(\mathbf{x})$. By setting the eigenvalues of the matrix $\mathbf{A}+\mathbf{B}\,\mathbf{F}$ equal to -1 to make the matrix Hurwitz, I obtain the matrix $\mathbf{F} = \begin{pmatrix} \frac{-3}{100} & \frac{597}{100} & \frac{-703}{100} & \frac{-1}{200} & \frac{11799}{200} & \frac{-9151}{200} \end{pmatrix}$.

I have used MAPLE to compute the trajectory of the system. I present the results here graphically.

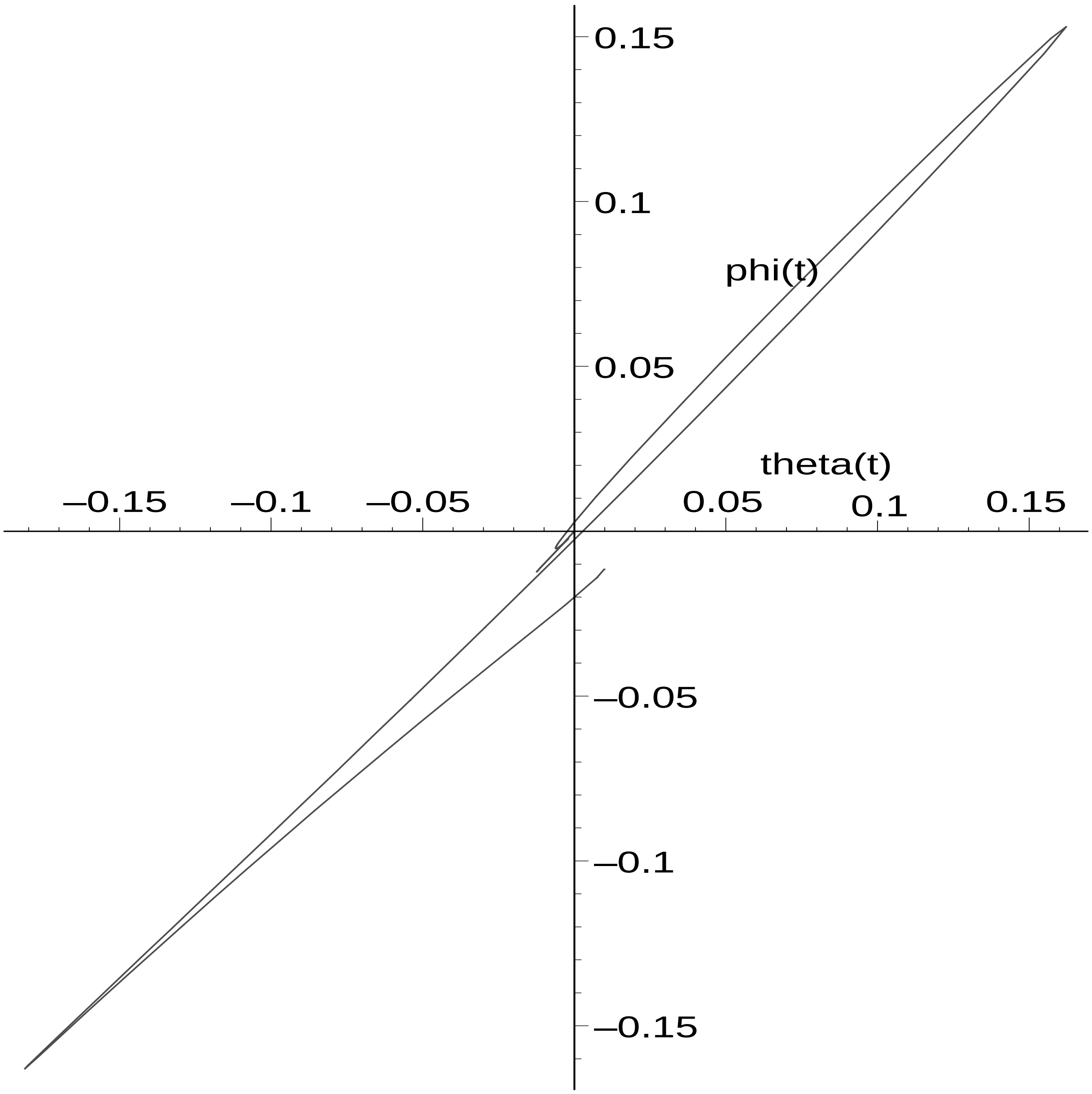


FIGURE C.1a

Figure C.1a shows the motion of the two pendulum bobs over time on a (θ, ϕ) axis.

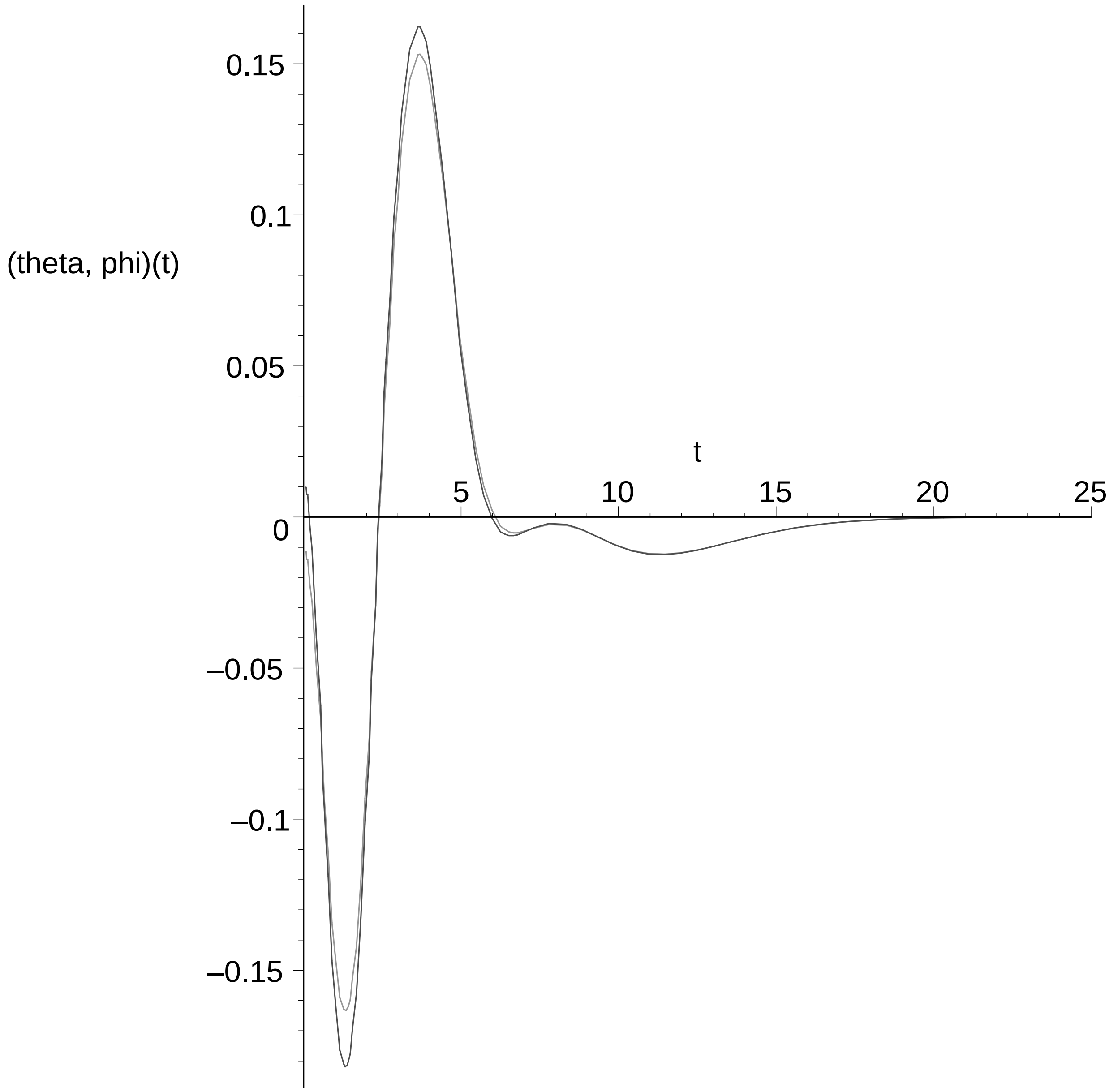


FIGURE C.1b

Figure C.1b also shows the motion of the two pendulum bobs over time. Figure C.1b shows the disposition of the two pendulum bobs over time as separate curves. Take a look back at Figure B.1b which shows the coefficients of g in accelerations of the two pendulum bobs. Observe that gravity accelerates the second pendulum bob towards $\phi = 0$ whenever $(\phi < \theta \, and \, \phi > 0) \, or \, (\phi > \theta \, and \, \phi < 0)$. In Figure C.1b, the darker curve represents $\theta(t)$ and the lighter curve represents

$\phi(t)$. Notice that ϕ is in fact, in this region most of the time. I find it absolutely astounding that a one-dimensional control with such a simple algorithm can control this system!

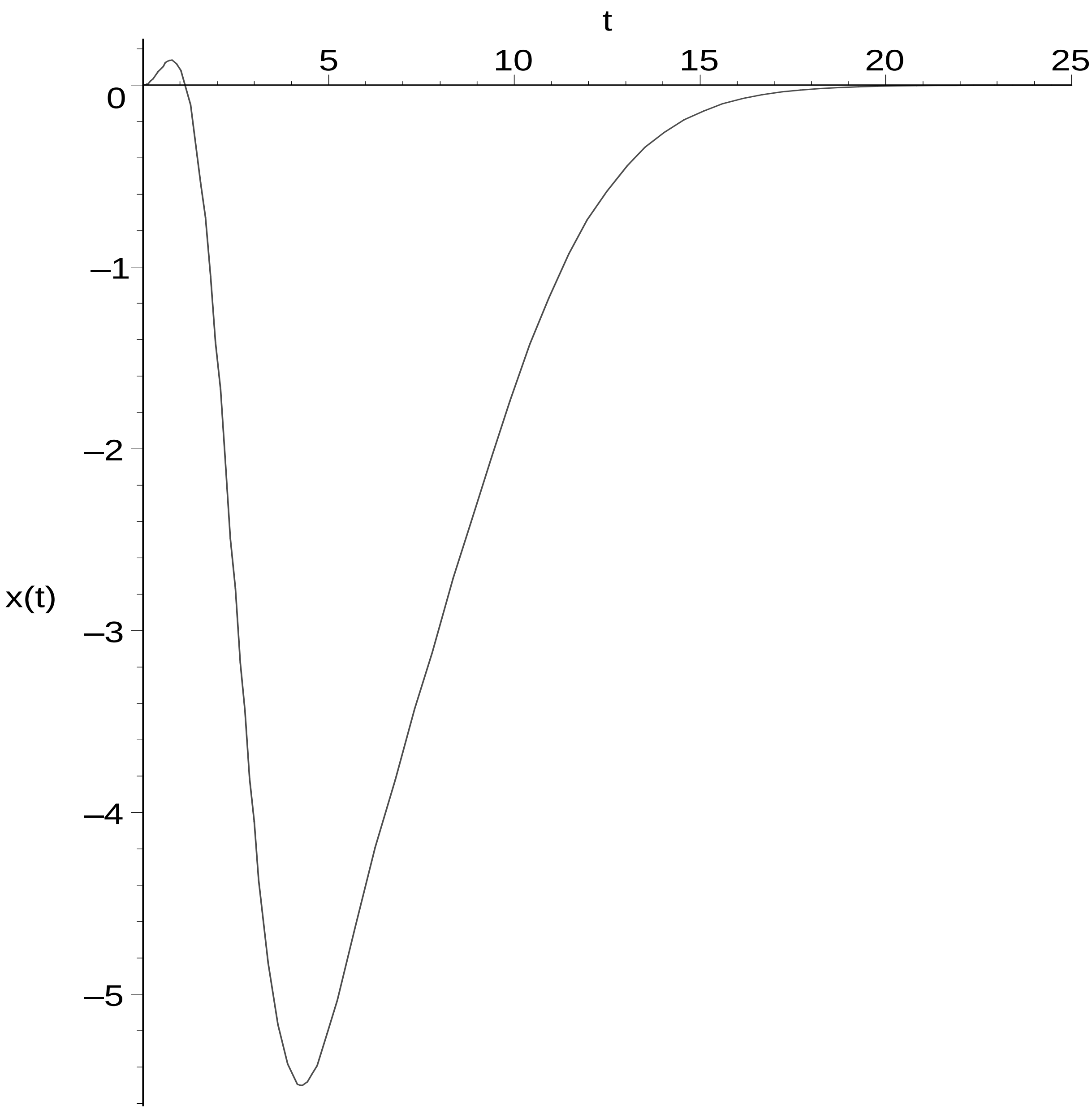


FIGURE C.2

Figure C.2 shows the motion of the cart over time. We see that the cart does move quite a bit as the two pendulum bobs are stabilized, but as I expected, it does come back to $x = 0$.

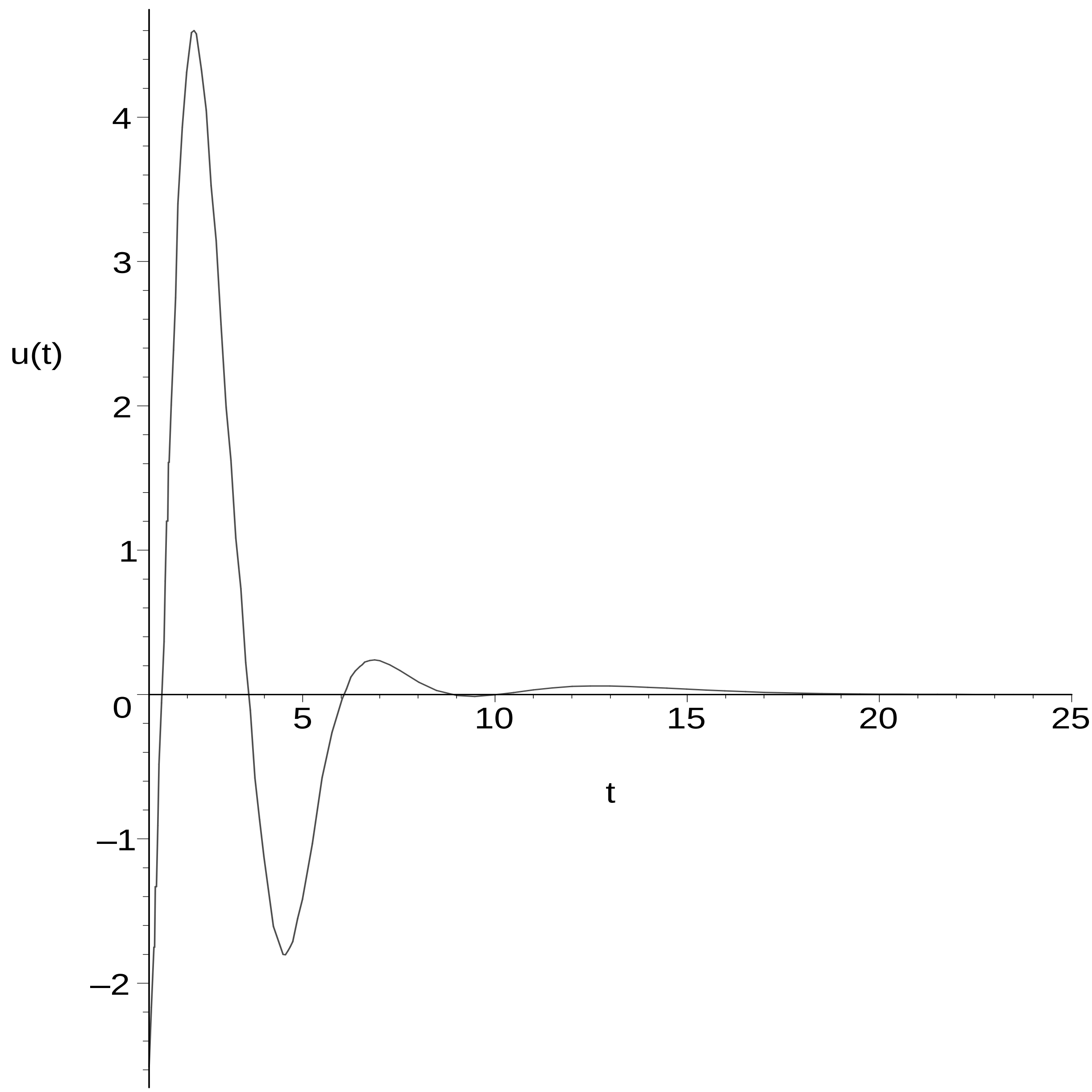


FIGURE C.3

Finally, here is a plot of the input force $u(t)$ over time. This plot calls our attention to questions such as if and how we could stabilize the system with limits on the input force. What if there are limits on the power output of the system? What if there are limits on how quickly the input force can be switched from one magnitude and direction to another? Even the seemingly simple question of how to stabilize a three dimensional system with a one dimensional controller raises many more questions.

13. Conclusion

Although I have only shown a single specific example of stabilization here, I have used MAPLE to do these calculations for many sample disturbances. There is little doubt in my mind that feedback is an excellent method for stabilizing a cart and double-pendulum system against small disturbances. As a matter of fact, according to [1], linear methods, such as stabilization have been used to build controllers to stabilize a real double-pendulum system. However, my work raises more questions than it answers. The most immediate is the question of how I can determine how big a disturbance this controller can stabilize against. Given that the only requirement for stability is that the matrix $(\mathbf{A} + \mathbf{B}\,\mathbf{F})$ be Hurwitz, how can we choose $\mathbf{F}$ to maximize the size of this neighborhood? How does all this change if I want to stabilize more than two pendulum bobs? I have calculated the equations, the theory guarantees stabilization just the same, and the literature says that engineers have built systems that stabilize not only three pendulum bobs, but four! [1]

I would like to conclude by answering the question, "What have I learned?" I learned quite a few things. I learned about a very efficient way to stabilize a cart and double-pendulum system. I learned about other methods in control theory that are appropriate for solving other problems, but perhaps not this one. For example, I tried unsuccessfully to stabilize the cart and double-pendulum by formulating stabilization as a problem in optimum control, a type of problem which assigns *cost* to each trajectory and asks what controller minimizes the cost. I've learned that although I can stabilize the system about the up unstable equilibrium (and also around the unstable equilibria with one pendulum arm pointing up and the other pointing down, which I did not describe in this paper), I do not understand the methods for steering from one unstable equilibrium to another; linearization does not hold much promise in my mind for solving this problem. I've learned that this problem has been extensively researched, and as interesting as my results may seem, they barely scratch the surface of what is known about the cart and double-pendulum, which itself is only one of many, many machines studied in control theory. Control theory is a field of intense research for a good reason: the theorems give us insight into how to control real systems.

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